Finiteness of the number of arithmetic groups generated by reflections in Lobachevsky spaces

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Abstract

After results by the author (1980, 1981) and Vinberg (1981), finiteness of the number of maximal arithmetic groups generated by reflections in Lobachevsky spaces was not known in dimensions $2 \le n \le 9$ only.

Recently (2005), the finiteness was proved in dimension 2 by Long, Maclachlan and Reid, and in dimension 3 by Agol.

Here we use these results in dimensions 2 and 3 to prove finiteness in all remaining dimensions $4 \le n \le 9$. Methods of the author (1980, 1981) are more than sufficient to prove this by a very short and very simple consideration.

1 Introduction

There are three types of simply-connected complete Riemannian manifolds of the constant curvature: spheres, Euclidean spaces and Lobachevsky (hyperbolic) spaces. Discrete reflection groups (i. e. generated by reflections in hyperplanes of these spaces) were defined by H.S.M. Coxeter. He classified these groups in spheres and Euclidean spaces. See [2].

There are two types of discrete reflection groups with fundamental domain of finite volume in Lobachevsky spaces: arithmetic and general. The subject of this paper is finiteness of the number of maximal arithmetic reflection groups in Lobachevsky spaces. We assume that the curvature is -1.

First, we remind known results about classification of arithmetic reflection groups in Lobachevsky spaces.

In [7], È.B. Vinberg (1967) gave the criterion of arithmeticity for discrete reflection groups in Lobachevsky spaces in terms of their fundamental chambers. In particular, he introduced the notion of the ground field of such group. This is a purely real algebraic number field of a finite degree over $\mathbf{Q}$. An arithmetic reflection group is a subgroup of finite index of the automorphism group of a hyperbolic quadratic form over the ring of integers of this field. Thus, there are two important integer parameters related to the arithmetic reflection group:

the dimension $n \geq 2$ of the Lobachevsky space, and the degree $N \geq 1$ of the ground field over $\mathbf{Q}$.

In [4], the author proved (1980) that the number of maximal arithmetic reflection groups is finite for the fixed parameters n and N. Moreover, the number of ground fields of the fixed degree N is also finite.

In [5], the author proved (1981) that there exists a constant N_0 such that, the dimension $n \leq 9$ if the degree $N \geq N_0$. Thus, the number of maximal arithmetic reflection groups is finite in each dimension $n \geq 10$ of Lobachevsky space.

In [8] and [9], È.B. Vinberg proved (1981) that the dimension $n < 30$. Thus, the number of maximal arithmetic reflection groups is finite in all dimensions $n \geq 10$.

Therefore, in the problem of finiteness of the number of arithmetic reflection groups in Lobachevsky spaces, only boundedness of the degree N of ground fields in dimensions $2 \leq n \leq 9$ of Lobachevsky spaces remained open.

See also reports [10] by È.B. Vinberg (1983) and [6] by the author (1986) in International Congresses of Mathematicians about these results.

During almost 25 years, there were no new general results in this domain. But, recently, some new important general results were obtained.

In [3], D.D. Long, C. Maclachlan, A.W. Reid proved (2005) that the number of maximal arithmetic reflection groups is finite in dimension $n = 2$. More generally, they proved finiteness of the number of maximal arithmetic Fuchsian groups of genus 0.

In [1], I. Agol proved (2005) that the number of maximal arithmetic reflection groups is finite in dimension $n = 3$.

Thus, after all these results, finiteness of the number of maximal arithmetic reflection groups is not proved in dimensions $4 \leq n \leq 9$ only.

The aim of this short paper is to show that the authors methods developed in [4] and [5] are more than sufficient to deduce this from finiteness results in dimensions $n = 2$ and $n = 3$. We don't use at all the very important part of the methods which is necessary to bound the dimension of the Lobachevsky space.

More exactly, we show that if the number of maximal arithmetic reflection groups is finite in dimensions 2 and 3, then from results of [4] and [5], it follows that the degree of ground fields in dimensions $4 \leq n \leq 9$ is bounded. By [4], it follows finiteness of the number of maximal arithmetic reflection groups in these dimensions.

Finally, this completes the proof of finiteness of the number of maximal arithmetic reflection groups in Lobachevsky spaces all together.

2 Finiteness of the number of maximal arithmetic reflection groups in Lobachevsky spaces of dimensions $4 \leq n \leq 9$

In the proof below, we work with the Klein model of Lobachevsky space $\mathcal{L}$ of dimension n which is related to a real hyperbolic form of the signature $(1, n)$. Then a finite convex polyhedron $\mathcal{M}$ in $\mathcal{L}$ is defined by the set $P(\mathcal{M})$ of vectors with the square -2 of this form which are perpendicular to hyperplanes $\mathcal{H}_\delta$, $\delta \in P(\mathcal{M})$, of faces of the highest (i. e. of the codimension one) dimension of the polyhedron $\mathcal{M}$ and directed outwards of $\mathcal{M}$. Then the number $\delta \cdot \delta'$ for $\delta, \delta' \in P(\mathcal{M})$ defines the angle or distance between the corresponding hyperplanes $\mathcal{H}_\delta$, $\mathcal{H}_{\delta'}$. In particular, these hyperplanes are perpendicular if and only if $\delta \cdot \delta' = 0$. The set $P(\mathcal{M})$ defines the Gram matrix $(\delta \cdot \delta')$, $\delta, \delta' \in P(\mathcal{M})$, of the polyhedron $\mathcal{M}$ and the equivalent Gram diagram $\Gamma(P(\mathcal{M}))$. It has vertices $P(\mathcal{M})$, and two different vertices δ, δ' are connected by the edge of the weight $\delta \cdot \delta'$ if $\delta \cdot \delta' \neq 0$. This defines connected components of $P(\mathcal{M})$ and of its subsets. See details in [7] and [4], [5].

The aim of this paper is to prove the following theorem.

Theorem 2.1. *In Lobachevsky spaces of dimensions $4 \leq n \leq 9$, the degree of ground fields of arithmetic reflection groups is bounded.*

Proof. We consider induction by the dimension n of Lobachevsky spaces.

For $n = 2$, the number of maximal arithmetic reflection groups is finite by [3]. Then, obviously, the degree of ground fields of arithmetic reflection groups is bounded in this dimension.

For $n = 3$, the number of maximal arithmetic reflection groups is finite by [1], and the degree of ground fields of arithmetic reflection groups is also bounded.

Let us assume that the degree of ground fields of arithmetic reflection groups is bounded in all dimensions $\leq n - 1$ of Lobachevsky spaces where $n \geq 4$. Let us prove this for n.

Let G be an arithmetic reflection group in Lobachevsky space of dimension n. Let $\mathcal{M}$ be a fundamental chamber of G. It is well-known that $\mathcal{M}$ is bounded (compact) if the ground field is different from $\mathbf{Q}$. Thus, we can assume that $\mathcal{M}$ is bounded. Let $P(\mathcal{M})$ be the set of all vectors with square -2 which are perpendicular to codimension one faces of $\mathcal{M}$ and directed outwards. We remind that $\mathcal{M}$ has acute angles, and then $\delta \cdot \delta' \geq 0$ for different $\delta, \delta' \in P(\mathcal{M})$.

Following [4] and [5], let us take a codimension one face $\mathcal{H}_e \cap \mathcal{M}$, $e \in P(\mathcal{M})$, of $\mathcal{M}$ which has minimality 14 (it defines the narrow part of $\mathcal{M}$) (see Sect. 2 in §4 of [5]). We denote by $P(\mathcal{M}, e)$ the set of all $\delta \in P(\mathcal{M})$ such that the hyperplane $\mathcal{H}_\delta$ intersects the hyperplane $\mathcal{H}_e$ (it is well-known that then $\mathcal{H}_e \cap \mathcal{H}_\delta \cap \mathcal{M}$ is a codimension two face of $\mathcal{M}$ if $\delta \neq e$). The minimality 14 of the face $\mathcal{H}_e \cap \mathcal{M}$ means that we have

$$\delta \cdot \delta' < 14, \quad \forall \delta, \delta' \in P(\mathcal{M}, e). \tag{2.1}$$

If $\mathcal{H}_\delta \perp \mathcal{H}_e$ (equivalently $\delta \cdot e = 0$) for all $\delta \in P(\mathcal{M}, e) - \{e\}$, then, obviously, the face $\mathcal{H}_e \cap \mathcal{M}$ is a fundamental chamber for an arithmetic reflection group in $\mathcal{H}_e$ of dimension $n-1$ with the same ground field. By induction hypothesis, it has bounded degree.

Thus, we can assume that there exists $f \in P(\mathcal{M}, e) - \{e\}$ such that $f \cdot e > 0$ (equivalently f and e are connected by an edge in the Gram diagram of $\mathcal{M}$). Let us consider the codimension two face $\mathcal{H}_e \cap \mathcal{H}_f \cap \mathcal{M}$ of $\mathcal{M}$. Let us denote by $P(\mathcal{M}, e, f)$ the set of all $\delta \in P(\mathcal{M})$ such that the hyperplane $\mathcal{H}_\delta$ intersects the codimension two subspace $\mathcal{H}_e \cap \mathcal{H}_f$ (it is well-known that then $\mathcal{H}_e \cap \mathcal{H}_f \cap \mathcal{H}_\delta \cap \mathcal{M}$ is a codimension 3 face of $\mathcal{M}$ if δ is different from e and f). If $\mathcal{H}_\delta \perp \mathcal{H}_e \cap \mathcal{H}_f$ (equivalently $\delta \cdot e = \delta \cdot f = 0$) for all $\delta \in P(\mathcal{M}, e, f) - \{e, f\}$, then, obviously, the codimension 2 face $\mathcal{H}_e \cap \mathcal{H}_f \cap \mathcal{M}$ of $\mathcal{M}$ is a fundamental chamber of arithmetic reflection group of dimension $n-2 \geq 2$ with the same ground field. By induction hypothesis, the field has bounded degree.

Thus, we can assume that there exists $g \in P(\mathcal{M}, e, f) - \{e, f\}$ such that $\mathcal{H}_g$ is not perpendicular to $\mathcal{H}_e \cap \mathcal{H}_f$. This means that either $g \cdot e > 0$ or $g \cdot f > 0$. Thus, the Gram diagram of e, f, g is connected, and their Gram matrix is negative definite.

Let us take a (one-dimensional) edge r in the face $\mathcal{H}_e \cap \mathcal{H}_f \cap \mathcal{M}$ of $\mathcal{M}$ which terminates in the hyperplane $\mathcal{H}_g$. Thus, one (of two) vertices of r is contained in $\mathcal{H}_g$, but another one is not contained (equivalently, the edge r is not contained in $\mathcal{H}_g$). Existence of such edge r is obvious.

Let $P(r)$ be the set of all $\delta \in P(\mathcal{M})$ such that $\mathcal{H}_\delta$ contains at least one of two vertices of the one-dimensional edge r. Since $r \subset \mathcal{H}_e \cap \mathcal{M}$, then $P(r) \subset P(\mathcal{M}, e)$, and $P(r)$ then defines the edge polyhedron of the minimality 14, since (2.1), according to the definition from (Sect. 2 of §2 in [5]). This edge polyhedron defines the ground field $\mathbf{K}$ of the group G. The elements $e, f, g \in P(r)$ have a connected Gram diagram, and their hyperplanes contain the same vertex of r, but the edge r is not contained in $\mathcal{H}_g$, by our construction. This shows that the hyperbolic (i. e. with hyperbolic Gram matrix) connected component of the Gram diagram $\Gamma(P(r))$ of this edge polyhedron is different from each of diagrams Γ_1, Γ_2 and Γ_3 of Theorem 2.3.1 from [5]. By the same Theorem 2.3.1 from [5], then $[\mathbf{K} : \mathbf{Q}]$ is bounded by the constant $N(14)$ of this theorem.

This finishes the proof. $\quad\square$

As we explained in Introduction, this completes the proof of finiteness of the number of maximal arithmetic reflection groups in Lobachevsky spaces all together.

Theorem 2.2. *The number of maximal arithmetic reflection groups in Lobachevsky spaces of dimensions $n \geq 2$ all together is finite.*

Their number is finite in each dimension $n \geq 4$ (the author). They don't exist in dimensions $n \geq 30$ (È.B. Vinberg). Their number is finite in the dimension $n = 2$ (D.D. Long, C. Maclachlan and A.W. Reid). Their number is finite in the dimension $n = 3$ (I. Agol).

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